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Coherent states of a charged particle in a uniform magnetic field

K Kowalski and J Rembieliński

Department of Theoretical Physics, University of Łódź, ul. Pomorska 149/153, 90-236 Łódź, Poland

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Abstract
The coherent states are constructed for a charged particle in a uniform magnetic field based on coherent states for the circular motion which have recently been introduced by the authors.

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1. Introduction

Coherent states which can be regarded from the physical point of view as the states closest to the classical ones are of fundamental importance in quantum physics. One of the most extensively studied quantum systems presented in many textbooks is a charged particle in a uniform magnetic field. The coherent states for this system were originally found by Malkin and Man'ko [1] (see also Feldman and Kahn [2]). As a matter of fact, the alternative states for a charged particle in a constant magnetic field were introduced by Loyola, Moshinsky and Szczepaniak [3] (see also the very recent paper by Schuch and Moshinsky [4]); nevertheless, those states are labelled by discrete quantum numbers and therefore can hardly be called 'coherent ones' which should be marked with the points of the classical phase space. In spite of the fact that the transverse motion of a charged particle in a uniform magnetic field is circular, the coherent states described by Malkin and Man'ko are related to the standard coherent states for a particle on a plane instead of the coherent states for a particle on a circle. Furthermore, the definition of the coherent states constructed by Malkin and Man'ko seems to ignore the momentum part of the classical phase space. In this work, we introduce the coherent states for a charged particle in a uniform magnetic field based on the construction of the coherent states for a quantum particle on a circle described in [5]. The paper is organized as follows. In section 2, we recall the construction of the coherent states for a particle on a circle. Section 3 summarizes the main facts about the quantization of a charged particle in a magnetic field. Section 4 is devoted to the definition of the coherent states for a charged particle in a magnetic field and the discussion of their most important properties. In section 5, we collect the basic facts about the coherent states for a charged particle in a magnetic field introduced by Malkin and Man'ko and we compare these states with ours discussed in section 4.

2. Coherent states for a quantum mechanics on a circle

In this section, we summarize the most important facts about the coherent states for a quantum particle on a circle. We first recall that the algebra adequate for the study of the motion on a circle is of the form

$$[J,U]=U,\qquad [J,U^\dagger]=-U^\dagger, \tag{2.1}$$

where J is the angular momentum operator, the unitary operator U represents the position of a quantum particle on a (unit) circle and we set $\hbar=1$. Consider the eigenvalue equation

$$J|j\rangle = j|j\rangle. \tag{2.2}$$

As shown in [5], j can only be an integer and half-integer. We restrict ourselves for brevity to the case of integer j. From (2.1) and (2.2), it follows that the operators U and $U^\dagger$ are the ladder operators, namely

$$U|j\rangle = |j+1\rangle,\qquad U^\dagger|j\rangle = |j-1\rangle. \tag{2.3}$$

Consider now the coherent states for a quantum particle on a circle. These states can be defined [5] as the solution of the eigenvalue equation

$$X|\xi\rangle = \xi|\xi\rangle, \tag{2.4}$$

where

$$X=\mathrm{e}^{-J+\frac{1}{2}}U. \tag{2.5}$$

An alternative construction of the coherent states specified by (2.4) based on the Weil–Brezin–Zak transform was described in [6]. The convenient parametrization of the complex number ξ consistent with the form of the operator X is given by

$$\xi=\mathrm{e}^{-l+\mathrm{i}\varphi}. \tag{2.6}$$

The parametrization (2.6) arises from the deformation of the cylinder (the phase space) specified by

$$x=\mathrm{e}^{-l}\cos\varphi,\qquad y=\mathrm{e}^{-l}\sin\varphi,\qquad z=l, \tag{2.7}$$

and then projecting the points of the obtained surface on the x, y plane. The projection of the vectors $|\xi\rangle$ onto the basis vectors $|j\rangle$ is of the form

$$\langle j|\xi\rangle = \xi^{-j}\,\mathrm{e}^{-\frac{j^2}{2}}. \tag{2.8}$$

Using the parameters l and φ, (2.8) can be written in the following equivalent form,

$$\langle j|l,\varphi\rangle = \mathrm{e}^{lj-\mathrm{i}j\varphi}\mathrm{e}^{-\frac{j^2}{2}}, \tag{2.9}$$

where $|l,\varphi\rangle \equiv |\xi\rangle$, with $\xi=\mathrm{e}^{-l+\mathrm{i}\varphi}$. The coherent states are not orthogonal. Namely,

$$\langle\xi|\eta\rangle = \sum_{j=-\infty}^{\infty}(\xi^*\eta)^{-j}\,\mathrm{e}^{-j^2} = \theta_3\left(\frac{\mathrm{i}}{2\pi}\ln\xi^*\eta\,\middle|\,\frac{\mathrm{i}}{\pi}\right), \tag{2.10}$$

where θ_3 is the Jacobi theta-function [7]. The coherent states satisfy

$$\frac{\langle l,\varphi|J|l,\varphi\rangle}{\langle l,\varphi|l,\varphi\rangle}\approx l, \tag{2.11}$$

where the maximal error is of order 0.1% and we have the exact equality in the case of l integer or half-integer. Therefore, the parameter l labelling the coherent states can be interpreted as the classical angular momentum. Furthermore, we have

$$\frac{\langle l,\varphi|U|l,\varphi\rangle}{\langle l,\varphi|l,\varphi\rangle}\approx \mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{\mathrm{i}\varphi}. \tag{2.12}$$

We point out that the absolute value of the average of the unitary operator U given by (2.12) which is approximately $\mathrm{e}^{-\frac{1}{4}}$ is lesser than 1, as expected because U is not diagonal in the coherent states basis. On introducing the relative expectation value

$$\langle\langle U\rangle\rangle_{(l,\varphi)} := \frac{\langle U\rangle_{(l,\varphi)}}{\langle U\rangle_{(0,0)}}, \tag{2.13}$$

where $\langle U\rangle_{(l,\varphi)} = \langle l,\varphi|U|l,\varphi\rangle/\langle l,\varphi|l,\varphi\rangle$, we get

$$\langle\langle U\rangle\rangle_{(l,\varphi)}\approx \mathrm{e}^{\mathrm{i}\varphi}. \tag{2.14}$$

Therefore, the relative expectation value $\langle\langle U\rangle\rangle_{(l,\varphi)}$ seems to be the most natural candidate to describe the average position on a circle and φ can be regarded as the classical angle. We finally point out that the discussed coherent states as well as the coherent states for a particle on a sphere introduced by us in [8] the are the concrete realization of the general mathematical scheme of construction of the Bargmann spaces described in the recent papers [9]. The importance of the coherent states for the circular motion has been confirmed by their recent application in quantum gravity [10].

3. Charged quantum particle in a magnetic field

In order to obtain the operators necessary for the definition of the coherent states, we first recall some facts about the quantization of a particle with mass μ and charge e in a uniform magnetic field $\boldsymbol{B} = (0,0,B)$, which is taken, without loss of generality, along the z-axis. Neglecting the spin we can write the Hamiltonian in the form

$$H = \frac{1}{2\mu}\boldsymbol{\pi}^2, \tag{3.1}$$

where $\boldsymbol{\pi} = \mu\dot{\boldsymbol{x}}$ is the kinetic momentum related to the canonical momentum $\boldsymbol{p}$ satisfying the Heisenberg algebra with the position $\boldsymbol{x}$, by

$$\boldsymbol{\pi} = \boldsymbol{p} - \mathrm{e}\boldsymbol{A}, \tag{3.2}$$

where $\boldsymbol{A}$ is the vector potential which fulfils $\boldsymbol{B} = \mathrm{rot}\,\boldsymbol{A}$ and we set $c = 1$. We choose the symmetric gauge such that

$$\boldsymbol{A} = (-By/2, Bx/2, 0) \tag{3.3}$$

in which $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}\times\boldsymbol{x}$. The coordinates of the kinetic momentum (3.2) in the gauge (3.3) are

$$\pi_x = p_x + \frac{\mu\omega}{2}y, \qquad \pi_y = p_y - \frac{\mu\omega}{2}x, \qquad \pi_z = p_z, \tag{3.4}$$

where $\omega = \frac{\mathrm{e}B}{\mu}$ is the cyclotron frequency. From (3.1) and (3.4), it follows that the motion along the z-axis is free and we actually deal with a two-dimensional problem in the x, y plane. Clearly, the Hamiltonian for the transverse motion is

$$H_\perp = \frac{1}{2\mu}\left(\pi_x^2 + \pi_y^2\right). \tag{3.5}$$

The coordinates π_x and π_y of the kinetic momentum given by (3.4) satisfy the following commutation relation:

$$[\pi_x, \pi_y] = \mathrm{i}\mu\omega, \tag{3.6}$$

where we set $\hbar = 1$. On introducing the operators

$$a = \frac{1}{\sqrt{2\mu\omega}}(-\pi_y + \mathrm{i}\pi_x), \qquad a^\dagger = \frac{1}{\sqrt{2\mu\omega}}(-\pi_y - \mathrm{i}\pi_x), \tag{3.7}$$

which obey

$$[a, a^\dagger] = 1, \tag{3.8}$$

we can write the Hamiltonian (3.5) in the form of the Hamiltonian of the harmonic oscillator, such that

$$H_\perp = \omega\left(a^\dagger a + \tfrac{1}{2}\right). \tag{3.9}$$

Consider now the orbit centre-coordinate operators [11]

$$x_0 = x + \frac{1}{\mu\omega}\pi_y, \qquad y_0 = y - \frac{1}{\mu\omega}\pi_x. \tag{3.10}$$

These operators are the integrals of the motion and they represent the coordinates of the centre of a circle in the x, y plane in which a particle moves. However, they do not commute with each other; namely, we have

$$[x_0, y_0] = -\frac{\mathrm{i}}{\mu\omega}. \tag{3.11}$$

As with the coordinates of the kinetic momentum, we can construct from x_0 and y_0 the creation and annihilation operators. We set

$$b = \sqrt{\frac{\mu\omega}{2}}(x_0 - \mathrm{i}y_0), \qquad b^\dagger = \sqrt{\frac{\mu\omega}{2}}(x_0 + \mathrm{i}y_0), \tag{3.12}$$

implying with the use of (3.11)

$$[b, b^\dagger] = 1. \tag{3.13}$$

We now return to (3.10). Since equation (3.10) holds also in the classical case, therefore the operators,

$$r_x := x - x_0 = -\frac{1}{\mu\omega}\pi_y, \qquad r_y := y - y_0 = \frac{1}{\mu\omega}\pi_x, \tag{3.14}$$

are the position observables of a particle on a circle. More precisely, they are the coordinates of the radius vector of a particle moving in a circle with the centre at the point (x_0, y_0). From (3.14) and (3.6), It follows that

$$[r_x, r_y] = \frac{\mathrm{i}}{\mu\omega}. \tag{3.15}$$

We have the formula on the squared radius of a circle such that

$$\boldsymbol{r}^2 = r_x^2 + r_y^2 = (x - x_0)^2 + (y - y_0)^2 = \frac{1}{(\mu\omega)^2}\left(\pi_x^2 + \pi_y^2\right) = \frac{2}{\mu\omega^2}H_\perp \tag{3.16}$$

following directly from (3.10) and (3.5).

4. Coherent states for a particle in a magnetic field

An experience with the coherent states for a circular motion described in section 2 indicates that in order to introduce the coherent states we should first identify the algebra adequate for the study of the motion of a charged particle in a uniform magnetic field. As with (2.1) such algebra should include the angular momentum operator. It seems that the most natural candidate is the operator defined by

$$L = (\boldsymbol{r} \times \boldsymbol{\pi})_z = r_x \pi_y - r_y \pi_x. \tag{4.1}$$

Indeed, equations (3.14) and (3.16) taken together yield

$$L = -\mu\omega \boldsymbol{r}^2, \tag{4.2}$$

which coincides with the classical expression. Furthermore, it can easily be verified that it commutes with the orbit centre-coordinate operators x_0 and y_0. It should be noted however that since

$$[L, r_x] = 2\mathrm{i} r_y, \qquad [L, r_y] = -2\mathrm{i} r_x, \tag{4.3}$$

following directly from (4.2) and (3.15), the generator of rotations about the axis passing through the centre of the circle and perpendicular to the x, y plane is not L but $\frac{1}{2}L$. Therefore, the counterpart of the operator J satisfying (2.1) which is the generator of the rotations is not L but $\frac{1}{2}L$.

Now, we introduce the operator representing the position of a particle on a circle of the form

$$r_+ = r_x + \mathrm{i} r_y. \tag{4.4}$$

This operator is a natural counterpart of the unitary operator U representing the position of a quantum particle on a unit circle discussed in section 2. Clearly, the algebra should include the orbit-centre operators x_0 and y_0. Bearing in mind the parametrization (4.4), it is plausible to introduce the operator

$$r_{0+} = x_0 + \mathrm{i} y_0 \tag{4.5}$$

which has the meaning of the operator corresponding to the centre of the circle. In order to complete the algebra, we also introduce the Hermitian conjugates of the operators r_+ and r_{0+}, respectively, such that

$$r_- = r_x - \mathrm{i} r_y, \qquad r_{0-} = x_0 - \mathrm{i} y_0. \tag{4.6}$$

Taking into account (4.3), (3.15) and (3.11), we arrive at the following algebra which seems to be most natural in the case of the circular motion of a charged particle in a uniform magnetic field:

$$\begin{aligned} &[L, r_\pm] = \pm 2 r_\pm, \qquad [L, r_{0\pm}] = 0, \qquad [r_+, r_-] = \frac{2}{\mu\omega}, \\ &[r_{0+}, r_{0-}] = -\frac{2}{\mu\omega}, \qquad [r_\pm, r_{0\pm}] = 0. \end{aligned} \tag{4.7}$$

The algebra (4.7) has the Casimir operator given in the unitary irreducible representation by

$$r_- r_+ + \frac{1}{\mu\omega} L = cI, \tag{4.8}$$

where c is a constant. We choose the representation referring to $c = -\frac{1}{\mu\omega}$ because it is the only one such that (4.8) with $r_\pm$ given by (4.4) and (4.6) is equivalent to (4.2). Consider now the creation and annihilation operators defined by

$$a = \sqrt{\frac{\mu\omega}{2}} r_+, \qquad a^\dagger = \sqrt{\frac{\mu\omega}{2}} r_-, \tag{4.9}$$

which coincide in view of (4.4) and (3.14) with the operators (3.7). The Casimir (4.8) with $c = -\frac{1}{m\omega}$ written with the help of the Bose operators (4.9) takes the form

$$L = -(2N_a + 1), \tag{4.10}$$

where $N_a = a^\dagger a$. Furthermore, it follows from (4.7) that the creation and annihilation operators such that (see (3.12), (4.5) and (4.6))

$$b = \sqrt{\frac{\mu\omega}{2}} r_{0-}, \qquad b^\dagger = \sqrt{\frac{\mu\omega}{2}} r_{0+} \tag{4.11}$$

commute with a and $a^\dagger$. Therefore, the operators $N_a = a^\dagger a$ and $N_b = b^\dagger b$ commute with each other. Consider the irreducible representation of the algebra (4.7) spanned by the common eigenvectors of the number operators N_a and N_b satisfying

$$N_a|n, m\rangle = n|n, m\rangle, \qquad N_b|n, m\rangle = m|n, m\rangle, \tag{4.12}$$

where n and m are non-negative integers. Using (4.10), (4.9) and (4.11), we find that the generators of the algebra (4.7) act on the basis vectors $|n, m\rangle$ in the following way:

$$L|n, m\rangle = -(2n + 1)|n, m\rangle, \tag{4.13a}$$

$$r_+|n, m\rangle = \sqrt{\frac{2n}{\mu\omega}}|n - 1, m\rangle, \tag{4.13b}$$

$$r_-|n, m\rangle = \sqrt{\frac{2(n + 1)}{\mu\omega}}|n + 1, m\rangle, \tag{4.13c}$$

$$r_{0+}|n, m\rangle = \sqrt{\frac{2(m + 1)}{\mu\omega}}|n, m + 1\rangle, \tag{4.13d}$$

$$r_{0-}|n, m\rangle = \sqrt{\frac{2m}{\mu\omega}}|n, m - 1\rangle. \tag{4.13e}$$

Now bearing in mind the form of the eigenvalue equation (2.4) and the discussion above, we define the coherent states for a charged particle in a uniform magnetic field as the simultaneous eigenvectors of the commuting non-Hermitian operators Z and r_{0-}:

$$Z|\zeta, z_0\rangle = \zeta|\zeta, z_0\rangle, \tag{4.14a}$$

$$r_{0-}|\zeta, z_0\rangle = z_0|\zeta, z_0\rangle, \tag{4.14b}$$

where

$$Z = \mathrm{e}^{-\frac{L}{2}+\frac{1}{2}} r_+, \tag{4.15}$$

and we recall that r_{0-} is proportional to the Bose annihilation operator b (see (4.11)), so that the coherent states $|\zeta, z_0\rangle$ can be viewed as the tensor product of the eigenvectors $|\zeta\rangle$ of the operator Z and the standard coherent states $|z_0\rangle$. Clearly, the complex number ζ parametrizes the classical phase space for the circular motion of a charged particle while the complex number z_0 represents the position of the centre of the circle. Taking into account (4.14) and (4.13), we find

$$\langle n, m|\zeta, z_0\rangle = \left(\frac{\mu\omega}{2}\right)^{\frac{n}{2}} \frac{\zeta^n}{\sqrt{n!}} \mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})^2} \left(\frac{\mu\omega}{2}\right)^{\frac{m}{2}} \frac{z_0^m}{\sqrt{m!}}. \tag{4.16}$$

Now, the form of the operator Z and (2.6) indicate the following parametrization of the complex number ζ:

$$\zeta = r(l)\,\mathrm{e}^{-\frac{l}{2}+\mathrm{i}\varphi}, \tag{4.17}$$

where l is real non-positive and $r(l) = \sqrt{-\frac{l}{\mu\omega}}$ is the classical radius of the circle in which moves a particle implied by the classical relation $l = -\mu\omega r^2$. Further, in accordance with (4.6) we set

$$z_0 = \overline{x}_0 - \mathrm{i}\overline{y}_0, \tag{4.18}$$

where $\overline{x}_0$ and $\overline{y}_0$ are real. Using (4.17) and (4.18), we can write (4.16) in the form

$$\langle n, m|l, \varphi; \overline{x}_0, \overline{y}_0\rangle = \left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^{\frac{n}{2}} \frac{\mathrm{e}^{\mathrm{i}n\varphi}}{\sqrt{n!}}\,\mathrm{e}^{-\frac{1}{2}(n+\frac{1}{2})^2} \frac{1}{\sqrt{m!}} \left[\frac{1}{\sqrt{2}}\left(\frac{\overline{x}_0}{\lambda} - \mathrm{i}\frac{\overline{y}_0}{\lambda}\right)\right]^m, \tag{4.19}$$

where $|l, \varphi; \overline{x}_0, \overline{y}_0\rangle \equiv |\zeta, z_0\rangle$ with ζ and z_0 given by (4.17) and (4.18), respectively, and $\lambda = 1/\sqrt{\mu\omega}$ is the classical radius of the ground-state Landau orbit.

As with the states $|\xi\rangle$ given by (2.4), our most important criterion to test the correctness of the introduced coherent states $|\zeta, z_0\rangle$ will be their closeness to the classical phase space. Consider the expectation value of the angular momentum operator L. Taking into account the completeness of the states $|n, m\rangle$, (4.13*a*) and (4.19) we get

$$\langle L\rangle_l = \frac{\langle l, \varphi; \overline{x}_0, \overline{y}_0|L|l, \varphi; \overline{x}_0, \overline{y}_0\rangle}{\langle l, \varphi; \overline{x}_0, \overline{y}_0|l, \varphi; \overline{x}_0, \overline{y}_0\rangle} = -\frac{\sum_{n=0}^{\infty} \frac{2n+1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2}}{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n \mathrm{e}^{-\left(n+\frac{1}{2}\right)^2}}. \tag{4.20}$$

From computer calculations, it follows that $\langle L\rangle_l \approx l$. Nevertheless, in opposition to the case of the coherent states for a quantum particle on a circle discussed in section 2, the approximate equality of $\langle L\rangle_l$ and l does not hold for practically arbitrary small $|l|$. More precisely, we have found that the approximation is very good for $|l| \geqslant 1$ (the bigger l the better approximation). For example if $|l| \sim 1$ then the relative error $|(\langle L\rangle_l - l)/l| \sim 1\%$. In our opinion such behaviour of $\langle L\rangle_l$ means that for small $|l|$ the quantum fluctuations are not negligible and the description based on the concept of the classical phase space is not an adequate one. We remark that the same phenomenon has been observed in the case of the coherent states for a particle on a sphere [8]. Thus, it turns out that the parameter l in (4.17) can be identified (in general approximately) with the classical angular momentum of a charged particle in a uniform magnetic field.

We now discuss the position of a particle on a circle in the context of the introduced coherent states. Using (4.13*b*) and (4.19), we find

$$\langle r_+\rangle_{(l,\varphi)} = \frac{\langle l, \varphi; \overline{x}_0, \overline{y}_0|r_+|l, \varphi; \overline{x}_0, \overline{y}_0\rangle}{\langle l, \varphi; \overline{x}_0, \overline{y}_0|l, \varphi; \overline{x}_0, \overline{y}_0\rangle} = r(l)\,\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{e}^{-\frac{1}{4}}\,\mathrm{e}^{-\frac{l}{2}} \frac{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n \mathrm{e}^{-(n+1)^2}}{\sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n \mathrm{e}^{-(n+\frac{1}{2})^2}}, \tag{4.21}$$

where $r(l) = \sqrt{-\frac{l}{\mu\omega}}$ is the classical formula on the radius of the circle in which a particle moves (see (4.17)). The computer calculations indicate that

$$\langle r_+\rangle_{(l,\varphi)} \approx r(l)\,\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{e}^{-\frac{1}{4}}, \tag{4.22}$$

where the approximation is very good but a bit worse than that in the case with $\langle L\rangle_l$. Namely, for $|l| = 5$ the relative error is of order 1%. Because of the term $\mathrm{e}^{-\frac{1}{4}}$, it turns out that the

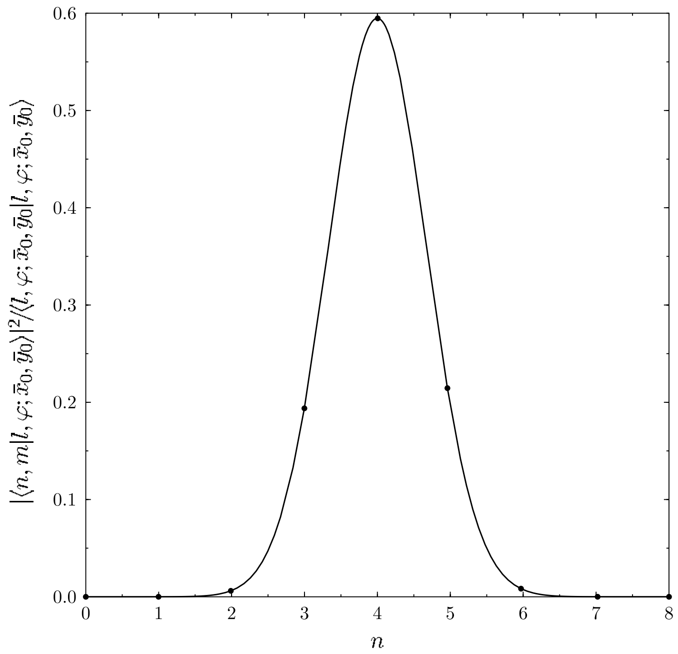

Figure 1. The plot of $p_{n,m}(l,\bar{x}_0,\bar{y}_0)$ versus n (see (4.25)), where $l=-9$, $m=0$ and $\bar{x}_0=\bar{y}_0=0$. The maximum is reached at point $n_{\max}=4$ coinciding with $-(l+1)/2$.

average value of r_+ does not belong to the circle with radius $r(l)$. Motivated by the formal resemblance of (4.22) with $r=1$ and (2.12), we identify the correct expectation value as

$$\langle\!\langle r_+\rangle\!\rangle_{(l,\varphi)} = \mathrm{e}^{\frac{1}{4}}\langle r_+\rangle_{(l,\varphi)} = r(l)\,\mathrm{e}^{\mathrm{i}\varphi}\,\mathrm{e}^{-\frac{l}{2}}\frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n\,\mathrm{e}^{-(n+1)^2}}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n\,\mathrm{e}^{-(n+\frac{1}{2})^2}}, \tag{4.23}$$

so

$$\langle\!\langle r_+\rangle\!\rangle_{(l,\varphi)} \approx r(l)\,\mathrm{e}^{\mathrm{i}\varphi} \tag{4.24}$$

which is a counterpart of (2.14). In our opinion, the appearance of the same factor $\mathrm{e}^{-\frac{1}{4}}$ in formulae (2.12) and (4.22) confirms the correctness of the approach taken up in this work. In view of the form of (4.24) it appears that $r(l)\,\mathrm{e}^{\mathrm{i}\varphi}$ (see 4.17) can be interpreted as the classical parametrization of a position of a charged particle in a uniform magnetic field.

We now study the distribution of vectors $|n,m\rangle$ in the normalized coherent state. The computer calculations indicate that the function

$$p_{n,m}(l,\overline{x}_0,\overline{y}_0) = \frac{|\langle n,m|l,\varphi;\overline{x}_0,\overline{y}_0\rangle|^2}{\langle l,\varphi;\overline{x}_0,\overline{y}_0|l,\varphi;\overline{x}_0,\overline{y}_0\rangle} = \frac{\frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n\mathrm{e}^{-(n+\frac{1}{2})^2}\frac{1}{m!}\left(\frac{\mu\omega}{2}\right)^m\left(\overline{x}_0^2+\overline{y}_0^2\right)^m}{\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{l}{2}\,\mathrm{e}^{-l}\right)^n\,\mathrm{e}^{-(n+\frac{1}{2})^2}\right)\mathrm{e}^{\frac{\mu\omega}{2}(\overline{x}_0^2+\overline{y}_0^2)}} \tag{4.25}$$

which gives the probability of finding the system in the state $|n,m\rangle$ when the system is in normalized coherent state $|l,\varphi;\overline{x}_0,\overline{y}_0\rangle/\sqrt{\langle l,\varphi;\overline{x}_0,\overline{y}_0|l,\varphi;\overline{x}_0,\overline{y}_0\rangle}$, is peaked for fixed l, m, x_0 and y_0 at point $n_{\max}$ coinciding with the integer nearest to $-(l+1)/2$ (see figure 1). In view of relation (4.10) this observation confirms once more the interpretation of the parameter l as the classical angular momentum. For the sake of completeness, we now write the formula on the expectation value of the operator r_{0-} representing the position of the centre of the circle, such that

$$\langle l, \varphi; \overline{x}_0, \overline{y}_0 | r_{0-} | l, \varphi; \overline{x}_0, \overline{y}_0 \rangle = \overline{x}_0 - \mathrm{i}\overline{y}_0 \tag{4.26}$$

following immediately from (4.14*b*) and (4.18). Thus, as expected $\overline{x}_0$ and $\overline{y}_0$ are the classical coordinates of the centre of the circle in which a particle moves.

We finally point out that the introduced coherent states are stable with respect to the evolution generated by the Hamiltonian $H_\perp$ given by (3.9). Indeed, we recall that x_0 and y_0, and thus r_{0-} are the integrals of the motion. Further equations (3.9) and (4.10) yield

$$H_\perp = -\omega \frac{L}{2}. \tag{4.27}$$

Hence, using (4.15) and the first commutator from (4.7) we get

$$Z(t) = \mathrm{e}^{\mathrm{i}tH_\perp} Z\, \mathrm{e}^{-\mathrm{i}tH_\perp} = \mathrm{e}^{-\mathrm{i}\omega t} Z \tag{4.28}$$

which leads to

$$Z(t)|\zeta, z_0\rangle = \zeta(t)|\zeta, z_0\rangle, \tag{4.29}$$

where $\zeta(t) = \mathrm{e}^{-\mathrm{i}\omega t}\zeta$.

5. Comparison with the Malkin–Man'ko coherent states

In this section, we compare the coherent states introduced above and the Malkin–Man'ko coherent states [1] mentioned in the introduction using as a test of correctness the closeness to the classical phase space. We first briefly sketch the basic properties of the Malkin–Man'ko coherent states. Up to an irrelevant muliplicative constant, these states can be defined as the common eigenvectors of the operators r_+ and r_{0-}

$$r_+|z, z_0\rangle = z|z, z_0\rangle, \tag{5.1a}$$

$$r_{0-}|z, z_0\rangle = z_0|z, z_0\rangle. \tag{5.1b}$$

Using (4.13*c*) and (4.13*d*), we find

$$\langle n, m|z, z_0\rangle = \left(\frac{\mu\omega}{2}\right)^{\frac{n}{2}} \frac{z^n}{\sqrt{n!}} \left(\frac{\mu\omega}{2}\right)^{\frac{m}{2}} \frac{z_0^m}{\sqrt{m!}}. \tag{5.2}$$

Of course, the states $|z, z_0\rangle$ are the standard coherent states for the Heisenberg–Weyl algebra generated by the operators $a, a^\dagger, b$ and $b^\dagger$ (see (4.9) and (4.11)). It is also clear that z and z_0 represent the position of a particle on a circle and the coordinates of the circle centre, respectively. The parametrization of the complex number z consistent with (4.4) is of the form

$$z = \overline{x} + \mathrm{i}\overline{y}, \tag{5.3}$$

where $\overline{x}$ and $\overline{y}$ are the rectangular coordinates of a particle on a circle. Evidently, the parametrization of z_0 is the same as in (4.18). Now, it follows directly from (5.1*a*) that

$$\langle r_+\rangle_{(\overline{x},\overline{y})} = \frac{\langle \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0 | r_+ | \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle}{\langle \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0 | \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle} = \overline{x} + \mathrm{i}\overline{y}, \tag{5.4}$$

where $|\overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle \equiv |z, z_0\rangle$ with z and z_0 given by (5.3) and (4.18), respectively. The corresponding formula on the expectation value of r_{0-} in the normalized coherent state $|\overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle / \sqrt{\langle \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0 | \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle}$ is the same as (4.26). Using the polar coordinates, we can write (5.4) in the form

$$\langle r_+\rangle^{\mathrm{MM}}_{(l,\varphi)} = \langle r_+\rangle_{(\overline{x},\overline{y})} = r(l)\,\mathrm{e}^{\mathrm{i}\varphi}, \tag{5.5}$$

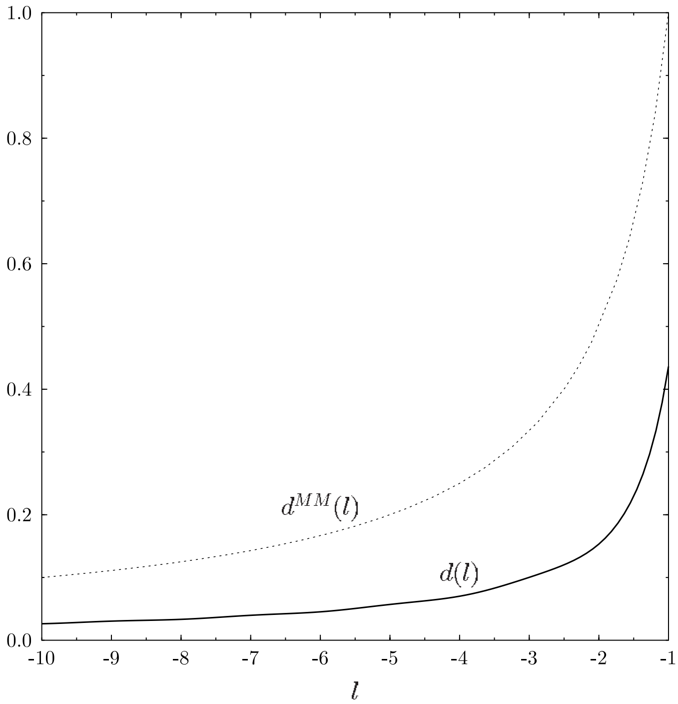

Figure 2. Comparison of the closeness to the phase space of the coherent states introduced in this work (solid line) and the Malkin–Man'ko coherent states (dotted line) by means of the distances $d(l)$ and $d^{\mathrm{MM}}(l)$ given by (5.8) and (5.9), respectively, with $\mu\omega = 1$.

where $r(l) = \sqrt{\overline{x}^2 + \overline{y}^2} = \sqrt{-\frac{l}{\mu\omega}}$ following from the classical formula $l = -\mu\omega r^2 = -\mu\omega(\overline{x}^2+\overline{y}^2)$; the indices MM are initials for Malkin–Man'ko. We point out that in opposition to (4.24) we have the exact relation (5.5). In this sense the Malkin–Man'ko coherent states are a better approximation of the configuration space than the states defined by us in the previous section. Furthermore, taking into account (4.8) with $c = -1/(\mu\omega)$, (5.1) and (5.3) we find

$$\langle L\rangle_{(\overline{x},\overline{y})} = \frac{\langle \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0|L|\overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle}{\langle \overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0|\overline{x}, \overline{y}; \overline{x}_0, \overline{y}_0\rangle} = -\mu\omega(\overline{x}^2 + \overline{y}^2) - 1. \tag{5.6}$$

Therefore, using the classical relation $l = -\mu\omega r^2 = -\mu\omega(\overline{x}^2 + \overline{y}^2)$, we get

$$\langle L\rangle_l^{\mathrm{MM}} = \langle L\rangle_{(\overline{x},\overline{y})} = l - 1. \tag{5.7}$$

Thus, it turns out that we have a shift in the classical momentum and the approximation $\langle L\rangle_l^{\mathrm{MM}} \approx l$ is worse in the light of the observations of section 4 (see discussion under formula (4.20)) than the approximate relation $\langle L\rangle_l \approx l$ which takes place in the case of the coherent states introduced in the previous section. In other words, the coherent states defined by (4.14) are better approximation of the 'momentum part' of the phase space. We stress that the shift in l in formula (5.7) is related to the zero-point energy and cannot be ignored. We finally remark that as with the states given by (4.14) the Malkin–Man'ko coherent states are stable with respect to the evolution generated by the Hamiltonian (3.9).

We now compare the coherent states discussed in section 4 and the coherent states introduced by Malkin and Man'ko taking as a criterion of correctness of the coherent states their closeness to the points of the classical phase space. Adopting the idea of the method of least squares, we use as the measure of such closeness the following entities,

$$d(l) = \sqrt{\left(\frac{\langle\langle r_+\rangle\rangle_{(l,0)} - r(l)}{r(l)}\right)^2 + \left(\frac{\langle L\rangle_l - l}{l}\right)^2}, \tag{5.8}$$

where $\langle\langle r_+\rangle\rangle_{(l,\varphi)}$ and $\langle L\rangle_l$ are given by (4.23) and (4.20), respectively, for the coherent states defined by (4.14), and analogously

$$d^{\mathrm{MM}}(l) = \sqrt{\left(\frac{\langle r_+\rangle^{\mathrm{MM}}_{(l,0)} - r(l)}{r(l)}\right)^2 + \left(\frac{\langle L\rangle^{\mathrm{MM}}_l - l}{l}\right)^2} = \frac{1}{|l|}, \tag{5.9}$$

for the Malkin–Man'ko coherent states, where in both the above formulae $r(l) = \sqrt{-\frac{l}{\mu\omega}}$ (see (4.23) and (5.5)). The distances $d(l)$ and $d^{\mathrm{MM}}(l)$ are compared in figure 2. As evident from figure 2, the coherent states for a charged particle in a magnetic field introduced in this paper are better approximations of the phase space than the coherent states of Malkin and Man'ko.

6. Conclusion

We have introduced in this work the new coherent states for a charged particle in a uniform magnetic field. The construction of these states based on the coherent states for the quantum mechanics on a circle seems to be more adequate than that of Malkin and Man'ko. Indeed, the fact that a classical particle moves transversely in a uniform magnetic field on a circle is recognized in the case of the Malkin–Man'ko coherent states only on the level of the evolution of these states. Furthermore, the coherent states introduced in this work are closer to the points of the classical phase space than the states discussed by Malkin and Man'ko. We realize that the best criterion for such closeness would be minimalization of some uncertainty relations. In the case of the coherent states for a particle on a circle the uncertainty relations have been introduced by authors in [12] (see also [13, 14]). Nevertheless, the problem of finding the analogous relations for the coherent states discussed herein seems to be a difficult task. The reason is that the radius of the circle is not a c-number as with the coherent states given by (2.4). Anyway, in our opinion, the simple criterion of closeness of the coherent states to the points of the classical phase space based on the definitions (5.8) and (5.9) is precise enough to decide that the coherent states introduced herein are better than those discovered by Malkin and Man'ko. Finally, the introduced coherent states should form a complete set. We recall that the completeness of coherent states is connected with the existence, via the 'resolution of the identity operator', of the Fock–Bargmann representation. However, the problem of finding the resolution of the identity operator is usually a nontrivial task. In our case, it is related to the solution of the problem of moments [15] such that

$$\int_0^\infty x^{n-1}\rho(x)\,\mathrm{d}x = n!\,\mathrm{e}^{(n+\frac{1}{2})^2},$$

where $\rho(x)$ is the unknown density. Because of the complexity of the problem the Fock–Bargmann representation for the introduced coherent states will be discussed in a separate work.

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